
Higher-Order Calculus of Variations on Time Scales

Rui A. C. Ferreira and Delfim F. M. Torres

Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal
ruiacferreira@yahoo.com, delfim@ua.pt

Summary. We prove a version of the Euler-Lagrange equations for certain problems of the calculus of variations on time scales with higher-order delta derivatives.

Keywords: time scales, calculus of variations, delta-derivatives of higher-order, Euler-Lagrange equations.

2000 Mathematics Subject Classification: 39A12, 49K05.

1 Introduction

Calculus of variations on time scales (we refer the reader to Section 2 for a brief introduction to time scales) has been introduced in 2004 in the papers by Bohner [2] and Hilscher and Zeidan [4], and seems to have many opportunities for application in economics [1]. In both works of Bohner and Hilscher&Zeidan, the Euler-Lagrange equation for the fundamental problem of the calculus of variations on time scales,

$$\mathcal{L}[y(\cdot)] = \int_a^b L(t, y^\sigma(t), y^\Delta(t)) \Delta t \longrightarrow \min, \quad y(a) = y_a, y(b) = y_b, \quad (1)$$

is obtained (in [4] for a bigger class of admissible functions and for problems with more general endpoint conditions). Here we generalize the previously obtained Euler-Lagrange equation for variational problems involving delta derivatives of more than the first order, i.e. for *higher-order problems*.

We consider the following extension to problem (1):

$$\begin{aligned}
\mathcal{L}[y(\cdot)] &= \int_a^{\rho^{r-1}(b)} L(t, y^{\sigma^r}(t), y^{\sigma^{r-1}\Delta}(t), \dots, y^{\sigma^{\Delta^{r-1}}}(t), y^{\Delta^r}(t)) \Delta t \longrightarrow \min, \\
y(a) &= y_a, \quad y(\rho^{r-1}(b)) = y_b, \\
&\vdots \\
y^{\Delta^{r-1}}(a) &= y_a^{r-1}, \quad y^{\Delta^{r-1}}(\rho^{r-1}(b)) = y_b^{r-1},
\end{aligned} \tag{P}$$

where $y^{\sigma^i \Delta^{r-i}}(t) \in \mathbb{R}^n$, $i \in \{0, \dots, r\}$, $n, r \in \mathbb{N}$, and t belongs to a time scale \mathbb{T} . Assumptions on the time scale \mathbb{T} are stated in Section 2; the conditions imposed on the Lagrangian L and on the admissible functions y are specified in Section 3. For $r = 1$ problem (P) is reduced to (1); for $\mathbb{T} = \mathbb{R}$ we get the classical problem of the calculus of variations with higher-order derivatives.

While in the classical context of the calculus of variations, i.e. when $\mathbb{T} = \mathbb{R}$, it is trivial to obtain the Euler-Lagrange necessary optimality condition for problem (P) as soon as we know how to do it for (1), this is not the case on the time scale setting. The Euler-Lagrange equation obtained in [2, 4] for (1) follow the classical proof, substituting the usual integration by parts formula by integration by parts for the delta integral (Lemma 2). Here we generalize the proof of [2, 4] to the higher-order case by successively applying the delta-integration by parts and thus obtaining a more general delta-differential Euler-Lagrange equation. It is worth to mention that such a generalization poses serious technical difficulties and that the obtained necessary optimality condition is not true on a general time scale, being necessary some restrictions on \mathbb{T} . Proving an Euler-Lagrange necessary optimality condition for a completely arbitrary time scale \mathbb{T} is a deep and difficult open question.

The paper is organized as follows: in Section 2 a brief introduction to the calculus of time scales is given and some assumptions and basic results provided. Then, under the assumed hypotheses on the time scale \mathbb{T} , we obtain in Section 3 the intended higher-order delta-differential Euler-Lagrange equation.

2 Basic Definitions and Results on Time Scales

A nonempty closed subset of \mathbb{R} is called a *time scale* and it is denoted by \mathbb{T} .

The *forward jump operator* $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\}, \text{ for all } t \in \mathbb{T},$$

while the *backward jump operator* $\rho : \mathbb{T} \rightarrow \mathbb{T}$ is defined by

$$\rho(t) = \sup \{s \in \mathbb{T} : s < t\}, \text{ for all } t \in \mathbb{T},$$

with $\inf \emptyset = \sup \mathbb{T}$ (i.e. $\sigma(M) = M$ if \mathbb{T} has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e. $\rho(m) = m$ if \mathbb{T} has a minimum m).

A point $t \in \mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense* and *left-scattered* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ and $\rho(t) < t$, respectively.

Throughout the paper we let $\mathbb{T} = [a, b] \cap \mathbb{T}_0$ with $a < b$ and \mathbb{T}_0 a time scale containing a and b .

Remark 1. The time scales \mathbb{T} considered in this work have a maximum b and, by definition, $\sigma(b) = b$. For example, let $[a, b] = [1, 5]$ and $\mathbb{T}_0 = \mathbb{N}$: in this case $\mathbb{T} = [1, 5] \cap \mathbb{N} = \{1, 2, 3, 4, 5\}$ and one has $\sigma(t) = t + 1$, $t \in \mathbb{T} \setminus \{5\}$, $\sigma(5) = 5$.

Following [3, pp. 2 and 11], we define $\mathbb{T}^k = \mathbb{T} \setminus (\rho(b), b]$, $\mathbb{T}^{k^2} = (\mathbb{T}^k)^k$ and, more generally, $\mathbb{T}^{k^n} = \left(\mathbb{T}^{k^{n-1}}\right)^k$, for $n \in \mathbb{N}$. The following standard notation is used for σ (and ρ): $\sigma^0(t) = t$, $\sigma^n(t) = (\sigma \circ \sigma^{n-1})(t)$, $n \in \mathbb{N}$.

The *graininess function* $\mu : \mathbb{T} \rightarrow [0, \infty)$ is defined by

$$\mu(t) = \sigma(t) - t, \text{ for all } t \in \mathbb{T}.$$

We say that a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is *delta differentiable* at $t \in \mathbb{T}^k$ if there is a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t (i.e. $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|, \text{ for all } s \in U.$$

We call $f^\Delta(t)$ the *delta derivative* of f at t .

If f is continuous at t and t is right-scattered, then (see Theorem 1.16 (ii) of [3])

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}. \quad (2)$$

Now, we define the r^{th} -delta derivative ($r \in \mathbb{N}$) of f to be the function $f^{\Delta^r} : \mathbb{T}^{k^r} \rightarrow \mathbb{R}$, provided $f^{\Delta^{r-1}}$ is delta differentiable on \mathbb{T}^{k^r} .

For delta differentiable functions f and g , the next formulas hold:

$$\begin{aligned} f^\sigma(t) &= f(t) + \mu(t)f^\Delta(t), \\ (fg)^\Delta(t) &= f^\Delta(t)g^\sigma(t) + f(t)g^\Delta(t) \\ &= f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t), \end{aligned} \quad (3)$$

where we abbreviate here and throughout $f \circ \sigma$ by f^σ . We will also write f^{Δ^σ} as f^{Δ^σ} and all the possible combinations of exponents of σ and Δ will be clear from the context.

The following lemma will be useful for our purposes.

Lemma 1. *Let $t \in \mathbb{T}^k$ ($t \neq \min \mathbb{T}$) satisfy the property $\rho(t) = t < \sigma(t)$. Then, the jump operator σ is not delta differentiable at t .*

Proof. We begin to prove that $\lim_{s \rightarrow t^-} \sigma(s) = t$. Let $\varepsilon > 0$ and take $\delta = \varepsilon$. Then for all $s \in (t - \delta, t)$ we have $|\sigma(s) - t| \leq |s - t| < \delta = \varepsilon$. Since $\sigma(t) > t$, this implies that σ is not continuous at t , hence not delta-differentiable by Theorem 1.16 (i) of [3]. \square

A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *rd-continuous* if it is continuous in right-dense points and if its left-sided limit exists in left-dense points. We denote the set of all rd-continuous functions by C_{rd} and the set of all differentiable functions with rd-continuous derivative by C_{rd}^1 .

It is known that rd-continuous functions possess an *antiderivative*, i.e. there exists a function F with $F^\Delta = f$, and in this case an *integral* is defined by $\int_a^b f(t)\Delta t = F(b) - F(a)$. It satisfies

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t). \quad (4)$$

We now present the integration by parts formulas for the delta integral:

Lemma 2. (*Theorem 1.77 (v) and (vi) of [3]*) If $a, b \in \mathbb{T}$ and $f, g \in C_{rd}^1$, then

1. $\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(t)\Delta t;$
2. $\int_a^b f(t)g^\Delta(t)\Delta t = [(fg)(t)]_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t.$

The main result of the calculus of variations on time scales for problem (1) is given by the following necessary optimality condition.

Theorem 1. ([2]) If $y_* \in C_{rd}^1$ is a weak local minimum of the problem

$$\mathcal{L}[y(\cdot)] = \int_a^b L(t, y^\sigma(t), y^\Delta(t))\Delta t \longrightarrow \min, \quad y(a) = y_a, \quad y(b) = y_b,$$

then the Euler-Lagrange equation

$$L_{y^\Delta}^\Delta(t, y_*^\sigma(t), y_*^\Delta(t)) = L_{y^\sigma}(t, y_*^\sigma(t), y_*^\Delta(t)),$$

$t \in \mathbb{T}^{k^2}$, holds.

Remark 2. In Theorem 1, and in what follows, the notation conforms to that used in [2]. Expression $L_{y^\Delta}^\Delta$ denotes the Δ derivative of a composition.

We will assume from now on that the time scale \mathbb{T} has sufficiently many points in order for all the calculations to make sense (with respect to this, we remark that Theorem 1 makes only sense if we are assuming a time scale \mathbb{T} with at least three points). Further, we consider time scales such that:

(H) $\sigma(t) = a_1 t + a_0$ for some $a_1 \in \mathbb{R}^+$ and $a_0 \in \mathbb{R}$, $t \in [a, b)$.

Under hypothesis (H) we have, among others, the differential calculus ($\mathbb{T}_0 = \mathbb{R}$, $a_1 = 1$, $a_0 = 0$), the difference calculus ($\mathbb{T}_0 = \mathbb{Z}$, $a_1 = a_0 = 1$) and the quantum calculus ($\mathbb{T}_0 = \{q^k : k \in \mathbb{N}_0\}$, with $q > 1$, $a_1 = q$, $a_0 = 0$).

Remark 3. From assumption (H) it follows by Lemma 1 that it is not possible to have points which are simultaneously left-dense and right-scattered. Also points that are simultaneously left-scattered and right-dense do not occur, since σ is strictly increasing.

Lemma 3. *Under hypothesis (H), if f is a two times delta differentiable function, then the next formula holds:*

$$f^{\sigma\Delta}(t) = a_1 f^{\Delta\sigma}(t), \quad t \in \mathbb{T}^{k^2}. \quad (5)$$

Proof. We have $f^{\sigma\Delta}(t) = [f(t) + \mu(t)f^{\Delta}(t)]^{\Delta}$ by formula (3). By the hypothesis on σ , μ is delta differentiable, hence $[f(t) + \mu(t)f^{\Delta}(t)]^{\Delta} = f^{\Delta}(t) + \mu^{\Delta}(t)f^{\Delta\sigma}(t) + \mu(t)f^{\Delta^2}(t)$ and applying again formula (3) we obtain $f^{\Delta}(t) + \mu^{\Delta}(t)f^{\Delta\sigma}(t) + \mu(t)f^{\Delta^2}(t) = f^{\Delta\sigma}(t) + \mu^{\Delta}(t)f^{\Delta\sigma}(t) = (1 + \mu^{\Delta}(t))f^{\Delta\sigma}(t)$. Now we only need to observe that $\mu^{\Delta}(t) = \sigma^{\Delta}(t) - 1$ and the result follows. \square

3 Main Results

Assume that the Lagrangian $L(t, u_0, u_1, \dots, u_r)$ of problem (P) has (standard) partial derivatives with respect to u_0, \dots, u_r , $r \geq 1$, and partial delta derivative with respect to t of order $r + 1$. Let $y \in C^{2r}$, where

$$C^{2r} = \left\{ y : \mathbb{T} \rightarrow \mathbb{R} : y^{\Delta^{2r}} \text{ is continuous on } \mathbb{T}^{k^{2r}} \right\}.$$

We say that $y_* \in C^{2r}$ is a *weak local minimum* for (P) provided there exists $\delta > 0$ such that $\mathcal{L}(y_*) \leq \mathcal{L}(y)$ for all $y \in C^{2r}$ satisfying the constraints in (P) and $\|y - y_*\|_{r,\infty} < \delta$, where

$$\|y\|_{r,\infty} := \sum_{i=0}^r \|y^{(i)}\|_{\infty},$$

with $y^{(i)} = y^{\sigma^i \Delta^{r-i}}$ and $\|y\|_{\infty} := \sup_{t \in \mathbb{T}^{k^r}} |y(t)|$.

Definition 1. *We say that $\eta \in C^{2r}$ is an admissible variation for problem (P) if*

$$\begin{aligned} \eta(a) &= 0, \quad \eta(\rho^{r-1}(b)) = 0 \\ &\vdots \\ \eta^{\Delta^{r-1}}(a) &= 0, \quad \eta^{\Delta^{r-1}}(\rho^{r-1}(b)) = 0. \end{aligned}$$

For simplicity of presentation, from now on we fix $r = 3$.

Lemma 4. *Suppose that f is defined on $[a, \rho^6(b)]$ and is continuous. Then, under hypothesis (H), $\int_a^{\rho^5(b)} f(t) \eta^{\sigma^3}(t) \Delta t = 0$ for every admissible variation η if and only if $f(t) = 0$ for all $t \in [a, \rho^6(b)]$.*

Proof. If $f(t) = 0$, then the result is obvious.

Now, suppose without loss of generality that there exists $t_0 \in [a, \rho^6(b)]$ such that $f(t_0) > 0$. First we consider the case in which t_0 is right-dense, hence left-dense or $t_0 = a$ (see Remark 3). If $t_0 = a$, then by the continuity of f at t_0 there exists a $\delta > 0$ such that for all $t \in [t_0, t_0 + \delta)$ we have $f(t) > 0$. Let us define η by

$$\eta(t) = \begin{cases} (t - t_0)^8(t - t_0 - \delta)^8 & \text{if } t \in [t_0, t_0 + \delta); \\ 0 & \text{otherwise.} \end{cases}$$

Clearly η is a C^6 function and satisfy the requirements of an admissible variation. But with this definition for η we get the contradiction

$$\int_a^{\rho^5(b)} f(t)\eta^{\sigma^3}(t)\Delta t = \int_{t_0}^{t_0+\delta} f(t)\eta^{\sigma^3}(t)\Delta t > 0.$$

Now, consider the case where $t_0 \neq a$. Again, the continuity of f ensures the existence of a $\delta > 0$ such that for all $t \in (t_0 - \delta, t_0 + \delta)$ we have $f(t) > 0$. Defining η by

$$\eta(t) = \begin{cases} (t - t_0 + \delta)^8(t - t_0 - \delta)^8 & \text{if } t \in (t_0 - \delta, t_0 + \delta); \\ 0 & \text{otherwise,} \end{cases}$$

and noting that it satisfy the properties of an admissible variation, we obtain

$$\int_a^{\rho^5(b)} f(t)\eta^{\sigma^3}(t)\Delta t = \int_{t_0-\delta}^{t_0+\delta} f(t)\eta^{\sigma^3}(t)\Delta t > 0,$$

which is again a contradiction.

Assume now that t_0 is right-scattered. In view of Remark 3, all the points t such that $t \geq t_0$ must be isolated. So, define η such that $\eta^{\sigma^3}(t_0) = 1$ and is zero elsewhere. It is easy to see that η satisfies all the requirements of an admissible variation. Further, using formula (4)

$$\int_a^{\rho^5(b)} f(t)\eta^{\sigma^3}(t)\Delta t = \int_{t_0}^{\sigma(t_0)} f(t)\eta^{\sigma^3}(t)\Delta t = \mu(t_0)f(t_0)\eta^{\sigma^3}(t_0) > 0,$$

which is a contradiction. \square

Theorem 2. *Let the Lagrangian $L(t, u_0, u_1, u_2, u_3)$ satisfy the conditions in the beginning of the section. On a time scale \mathbb{T} satisfying (H), if y_* is a weak local minimum for the problem of minimizing*

$$\int_a^{\rho^2(b)} L\left(t, y^{\sigma^3}(t), y^{\sigma^2\Delta}(t), y^{\sigma\Delta^2}(t), y^{\Delta^3}(t)\right) \Delta t$$

subject to

$$\begin{aligned}
y(a) &= y_a, \quad y(\rho^2(b)) = y_b, \\
y^\Delta(a) &= y_a^1, \quad y^\Delta(\rho^2(b)) = y_b^1, \\
y^{\Delta^2}(a) &= y_a^2, \quad y^{\Delta^2}(\rho^2(b)) = y_b^2,
\end{aligned}$$

then y_* satisfies the Euler-Lagrange equation

$$L_{u_0}(\cdot) - L_{u_1}^\Delta(\cdot) + \frac{1}{a_1} L_{u_2}^{\Delta^2}(\cdot) - \frac{1}{a_1^3} L_{u_3}^{\Delta^3}(\cdot) = 0, \quad t \in [a, \rho^6(b)],$$

where $(\cdot) = (t, y_*^{\sigma^3}(t), y_*^{\sigma^2\Delta}(t), y_*^{\sigma\Delta^2}(t), y_*^{\Delta^3}(t))$.

Proof. Suppose that y_* is a weak local minimum of \mathcal{L} . Let $\eta \in C^6$ be an admissible variation, i.e. η is an arbitrary function such that η , η^Δ and η^{Δ^2} vanish at $t = a$ and $t = \rho^2(b)$. Define function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ by $\Phi(\varepsilon) = \mathcal{L}(y_* + \varepsilon\eta)$. This function has a minimum at $\varepsilon = 0$, so we must have (see [2, Theorem 3.2])

$$\Phi'(0) = 0. \quad (6)$$

Differentiating Φ under the integral sign (we can do this in virtue of the conditions we imposed on L) with respect to ε and setting $\varepsilon = 0$, we obtain from (6) that

$$\begin{aligned}
0 = \int_a^{\rho^2(b)} \left\{ L_{u_0}(\cdot) \eta^{\sigma^3}(t) + L_{u_1}(\cdot) \eta^{\sigma^2\Delta}(t) \right. \\
\left. + L_{u_2}(\cdot) \eta^{\sigma\Delta^2}(t) + L_{u_3}(\cdot) \eta^{\Delta^3}(t) \right\} \Delta t. \quad (7)
\end{aligned}$$

Since we will delta differentiate L_{u_i} , $i = 1, 2, 3$, we rewrite (7) in the following form:

$$\begin{aligned}
0 = \int_a^{\rho^3(b)} \left\{ L_{u_0}(\cdot) \eta^{\sigma^3}(t) + L_{u_1}(\cdot) \eta^{\sigma^2\Delta}(t) \right. \\
\left. + L_{u_2}(\cdot) \eta^{\sigma\Delta^2}(t) + L_{u_3}(\cdot) \eta^{\Delta^3}(t) \right\} \Delta t \\
+ \mu(\rho^3(b)) \left\{ L_{u_0} \eta^{\sigma^3} + L_{u_1} \eta^{\sigma^2\Delta} + L_{u_2} \eta^{\sigma\Delta^2} + L_{u_3} \eta^{\Delta^3} \right\} (\rho^3(b)). \quad (8)
\end{aligned}$$

Integrating (8) by parts gives

$$\begin{aligned}
0 = \int_a^{\rho^3(b)} \left\{ L_{u_0}(\cdot) \eta^{\sigma^3}(t) - L_{u_1}^\Delta(\cdot) \eta^{\sigma^3}(t) \right. \\
\left. - L_{u_2}^\Delta(\cdot) \eta^{\sigma\Delta^2}(t) - L_{u_3}^\Delta(\cdot) \eta^{\Delta^2\sigma}(t) \right\} \Delta t \\
+ \left[L_{u_1}(\cdot) \eta^{\sigma^2}(t) \right]_{t=a}^{t=\rho^3(b)} + \left[L_{u_2}(\cdot) \eta^{\sigma\Delta}(t) \right]_{t=a}^{t=\rho^3(b)} + \left[L_{u_3}(\cdot) \eta^{\Delta^2}(t) \right]_{t=a}^{t=\rho^3(b)} \\
+ \mu(\rho^3(b)) \left\{ L_{u_0} \eta^{\sigma^3} + L_{u_1} \eta^{\sigma^2\Delta} + L_{u_2} \eta^{\sigma\Delta^2} + L_{u_3} \eta^{\Delta^3} \right\} (\rho^3(b)). \quad (9)
\end{aligned}$$

Now we show how to simplify (9). We start by evaluating $\eta^{\sigma^2}(a)$:

$$\begin{aligned}
\eta^{\sigma^2}(a) &= \eta^\sigma(a) + \mu(a)\eta^{\sigma\Delta}(a) \\
&= \eta(a) + \mu(a)\eta^\Delta(a) + \mu(a)a_1\eta^{\Delta\sigma}(a) \\
&= \mu(a)a_1\left(\eta^\Delta(a) + \mu(a)\eta^{\Delta^2}(a)\right) \\
&= 0,
\end{aligned} \tag{10}$$

where the last term of (10) follows from (5). Now, we calculate $\eta^{\sigma\Delta}(a)$. By (5) we have $\eta^{\sigma\Delta}(a) = a_1\eta^{\Delta\sigma}(a)$ and applying (3) we obtain

$$a_1\eta^{\Delta\sigma}(a) = a_1\left(\eta^\Delta(a) + \mu(a)\eta^{\Delta^2}(a)\right) = 0.$$

Now we turn to analyze what happens at $t = \rho^3(b)$. It is easy to see that if b is left-dense, then the last terms of (9) vanish. So suppose that b is left-scattered. Since σ is delta differentiable, by Lemma 1 we cannot have points which are simultaneously left-dense and right-scattered. Hence, $\rho(b)$, $\rho^2(b)$ and $\rho^3(b)$ are right-scattered points. Now, by hypothesis $\eta^\Delta(\rho^2(b)) = 0$, hence we have by (2) that

$$\frac{\eta(\rho(b)) - \eta(\rho^2(b))}{\mu(\rho^2(b))} = 0.$$

But $\eta(\rho^2(b)) = 0$, hence $\eta(\rho(b)) = 0$. Analogously, we have

$$\eta^{\Delta^2}(\rho^2(b)) = 0 \Leftrightarrow \frac{\eta^\Delta(\rho(b)) - \eta^\Delta(\rho^2(b))}{\mu(\rho^2(b))} = 0,$$

from what follows that $\eta^\Delta(\rho(b)) = 0$. This last equality implies $\eta(b) = 0$. Applying previous expressions to the last terms of (9), we obtain:

$$\begin{aligned}
\eta^{\sigma^2}(\rho^3(b)) &= \eta(\rho(b)) = 0, \\
\eta^{\sigma\Delta}(\rho^3(b)) &= \frac{\eta^{\sigma^2}(\rho^3(b)) - \eta^\sigma(\rho^3(b))}{\mu(\rho^3(b))} = 0, \\
\eta^{\sigma^3}(\rho^3(b)) &= \eta(b) = 0, \\
\eta^{\sigma^2\Delta}(\rho^3(b)) &= \frac{\eta^{\sigma^3}(\rho^3(b)) - \eta^{\sigma^2}(\rho^3(b))}{\mu(\rho^3(b))} = 0, \\
\eta^{\sigma\Delta^2}(\rho^3(b)) &= \frac{\eta^{\sigma\Delta}(\rho^2(b)) - \eta^{\sigma\Delta}(\rho^3(b))}{\mu(\rho^3(b))} \\
&= \frac{\frac{\eta^\sigma(\rho(b)) - \eta^\sigma(\rho^2(b))}{\mu(\rho^2(b))} - \frac{\eta^\sigma(\rho^2(b)) - \eta^\sigma(\rho^3(b))}{\mu(\rho^3(b))}}{\mu(\rho^3(b))} \\
&= 0.
\end{aligned}$$

In view of our previous calculations,

$$\begin{aligned} & \left[L_{u_1}(\cdot) \eta^{\sigma^2}(t) \right]_{t=a}^{t=\rho^3(b)} + \left[L_{u_2}(\cdot) \eta^{\sigma \Delta}(t) \right]_{t=a}^{t=\rho^3(b)} + \left[L_{u_3}(\cdot) \eta^{\Delta^2}(t) \right]_{t=a}^{t=\rho^3(b)} \\ & + \mu(\rho^3(b)) \left\{ L_{u_0} \eta^{\sigma^3} + L_{u_1} \eta^{\sigma^2 \Delta} + L_{u_2} \eta^{\sigma \Delta^2} + L_{u_3} \eta^{\Delta^3} \right\} (\rho^3(b)) \end{aligned}$$

is reduced to¹

$$L_{u_3}(\rho^3(b)) \eta^{\Delta^2}(\rho^3(b)) + \mu(\rho^3(b)) L_{u_3}(\rho^3(b)) \eta^{\Delta^3}(\rho^3(b)). \quad (11)$$

Now note that

$$\eta^{\Delta^2 \sigma}(\rho^3(b)) = \eta^{\Delta^2}(\rho^3(b)) + \mu(\rho^3(b)) \eta^{\Delta^3}(\rho^3(b))$$

and by hypothesis $\eta^{\Delta^2 \sigma}(\rho^3(b)) = \eta^{\Delta^2}(\rho^2(b)) = 0$. Therefore,

$$\mu(\rho^3(b)) \eta^{\Delta^3}(\rho^3(b)) = -\eta^{\Delta^2}(\rho^3(b)),$$

from which follows that (11) must be zero. We have just simplified (9) to

$$\begin{aligned} 0 = \int_a^{\rho^3(b)} & \left\{ L_{u_0}(\cdot) \eta^{\sigma^3}(t) - L_{u_1}^{\Delta}(\cdot) \eta^{\sigma^3}(t) \right. \\ & \left. - L_{u_2}^{\Delta}(\cdot) \eta^{\sigma \Delta \sigma}(t) - L_{u_3}^{\Delta}(\cdot) \eta^{\Delta^2 \sigma}(t) \right\} \Delta t. \quad (12) \end{aligned}$$

In order to apply again the integration by parts formula, we must first make some transformations in $\eta^{\sigma \Delta \sigma}$ and $\eta^{\Delta^2 \sigma}$. By (5) we have

$$\eta^{\sigma \Delta \sigma}(t) = \frac{1}{a_1} \eta^{\sigma^2 \Delta}(t) \quad (13)$$

and

$$\eta^{\Delta^2 \sigma}(t) = \frac{1}{a_1^2} \eta^{\sigma \Delta^2}(t). \quad (14)$$

Hence, (12) becomes

$$\begin{aligned} 0 = \int_a^{\rho^3(b)} & \left\{ L_{u_0}(\cdot) \eta^{\sigma^3}(t) - L_{u_1}^{\Delta}(\cdot) \eta^{\sigma^3}(t) \right. \\ & \left. - \frac{1}{a_1} L_{u_2}^{\Delta}(\cdot) \eta^{\sigma^2 \Delta}(t) - \frac{1}{a_1^2} L_{u_3}^{\Delta}(\cdot) \eta^{\sigma \Delta^2}(t) \right\} \Delta t. \quad (15) \end{aligned}$$

By the same reasoning as before, (15) is equivalent to

¹ In what follows there is some abuse of notation: $L_{u_3}(\rho^3(b))$ denotes $L_{u_3}(\cdot)|_{t=\rho^3(b)}$, that is, we substitute t in $(\cdot) = (t, y_*^{\sigma^3}(t), y_*^{\sigma^2 \Delta}(t), y_*^{\sigma \Delta^2}(t), y_*^{\Delta^3}(t))$ by $\rho^3(b)$.

$$\begin{aligned}
0 = & \int_a^{\rho^4(b)} \left\{ L_{u_0}(\cdot) \eta^{\sigma^3}(t) - L_{u_1}^\Delta(\cdot) \eta^{\sigma^3}(t) \right. \\
& \left. - \frac{1}{a_1} L_{u_2}^\Delta(\cdot) \eta^{\sigma^2 \Delta}(t) - \frac{1}{a_1^2} L_{u_3}^\Delta(\cdot) \eta^{\sigma \Delta^2}(t) \right\} \Delta t \\
& + \mu(\rho^4(b)) \left\{ L_{u_0} \eta^{\sigma^3} - L_{u_1}^\Delta \eta^{\sigma^3} - \frac{1}{a_1} L_{u_2}^\Delta \eta^{\sigma^2 \Delta} - \frac{1}{a_1^2} L_{u_3}^\Delta \eta^{\sigma \Delta^2} \right\} (\rho^4(b))
\end{aligned}$$

and integrating by parts we obtain

$$\begin{aligned}
0 = & \int_a^{\rho^4(b)} \left\{ L_{u_0}(\cdot) \eta^{\sigma^3}(t) - L_{u_1}^\Delta(\cdot) \eta^{\sigma^3}(t) \right. \\
& \left. + \frac{1}{a_1} L_{u_2}^{\Delta^2}(\cdot) \eta^{\sigma^3}(t) + \frac{1}{a_1^2} L_{u_3}^{\Delta^2}(\cdot) \eta^{\sigma \Delta^2}(t) \right\} \Delta t \\
& - \left[\frac{1}{a_1} L_{u_2}^\Delta(\cdot) \eta^{\sigma^2}(t) \right]_{t=a}^{t=\rho^4(b)} - \left[\frac{1}{a_1^2} L_{u_3}^\Delta(\cdot) \eta^{\sigma \Delta}(t) \right]_{t=a}^{t=\rho^4(b)} \\
& + \mu(\rho^4(b)) \left\{ L_{u_0} \eta^{\sigma^3} - L_{u_1}^\Delta \eta^{\sigma^3} - \frac{1}{a_1} L_{u_2}^\Delta \eta^{\sigma^2 \Delta} - \frac{1}{a_1^2} L_{u_3}^\Delta \eta^{\sigma \Delta^2} \right\} (\rho^4(b)).
\end{aligned} \tag{16}$$

Using analogous arguments to those above, we simplify (16) to

$$\begin{aligned}
& \int_a^{\rho^4(b)} \left\{ L_{u_0}(\cdot) \eta^{\sigma^3}(t) - L_{u_1}^\Delta(\cdot) \eta^{\sigma^3}(t) \right. \\
& \left. + \frac{1}{a_1} L_{u_2}^\Delta(\cdot) \eta^{\sigma^2 \Delta}(t) + \frac{1}{a_1^3} L_{u_3}^{\Delta^2}(\cdot) \eta^{\sigma^2 \Delta}(t) \right\} \Delta t = 0.
\end{aligned}$$

Calculations as done before lead us to the final expression

$$\begin{aligned}
& \int_a^{\rho^5(b)} \left\{ L_{u_0}(\cdot) \eta^{\sigma^3}(t) - L_{u_1}^\Delta(\cdot) \eta^{\sigma^3}(t) \right. \\
& \left. + \frac{1}{a_1} L_{u_2}^{\Delta^2}(\cdot) \eta^{\sigma^3}(t) - \frac{1}{a_1^3} L_{u_3}^{\Delta^3}(\cdot) \eta^{\sigma^3}(t) \right\} \Delta t = 0,
\end{aligned}$$

which is equivalent to

$$\int_a^{\rho^5(b)} \left\{ L_{u_0}(\cdot) - L_{u_1}^\Delta(\cdot) + \frac{1}{a_1} L_{u_2}^{\Delta^2}(\cdot) - \frac{1}{a_1^3} L_{u_3}^{\Delta^3}(\cdot) \right\} \eta^{\sigma^3}(t) \Delta t = 0. \tag{17}$$

Applying Lemma 4 to (17), we obtain the Euler-Lagrange equation

$$L_{u_0}(\cdot) - L_{u_1}^\Delta(\cdot) + \frac{1}{a_1} L_{u_2}^{\Delta^2}(\cdot) - \frac{1}{a_1^3} L_{u_3}^{\Delta^3}(\cdot) = 0, \quad t \in [a, \rho^6(b)].$$

□

Following exactly the same steps in the proofs of Lemma 4 and Theorem 2 for an arbitrary $r \in \mathbb{N}$, one easily obtains the Euler-Lagrange equation for problem (P).

Theorem 3. *(Necessary optimality condition for problems of the calculus of variations with higher-order delta derivatives) On a time scale \mathbb{T} satisfying hypothesis (H), if y_* is a weak local minimum for problem (P), then y_* satisfies the Euler-Lagrange equation*

$$\sum_{i=0}^r (-1)^i \left(\frac{1}{a_1} \right)^{\frac{(i-1)i}{2}} L_{u_i}^{\Delta^i} \left(t, y_*^{\sigma^r}(t), y_*^{\sigma^{r-1}\Delta}(t), \dots, y_*^{\sigma^{\Delta^{r-1}}}(t), y_*^{\Delta^r}(t) \right) = 0, \quad (18)$$

$$t \in [a, \rho^{2r}(b)].$$

Remark 4. The factor $\left(\frac{1}{a_1} \right)^{\frac{(i-1)i}{2}}$ in (18) comes from the fact that, after each time we apply the integration by parts formula, we commute successively σ with Δ using (5) (see formulas (13) and (14)), doing this $\sum_{j=1}^{i-1} j = \frac{(i-1)i}{2}$ times for each of the parcels within the integral.

Acknowledgments

This work is part of the first author's PhD project and was partially supported by the Control Theory Group (cotg) of the Centre for Research on Optimization and Control (CEOC), through the Portuguese Foundation for Science and Technology (FCT), cofinanced by the European Community Fund FEDER/POCI 2010. The authors are grateful to Dorota Mozyrska and Ewa Pawluszewicz for inspiring discussions during the Workshop on Mathematical Control Theory and Finance, Lisbon, 10-14 April 2007, where some preliminary results were presented; to an anonymous referee for helpful comments.

References

1. Atici FM, Biles DC, Lebedinsky A (2006) An application of time scales to economics. *Math. Comput. Modelling* 43(7-8):718–726
2. Bohner M (2004) Calculus of variations on time scales. *Dyn. Sys. and Appl.* 272(13):339–349
3. Bohner M, Peterson A (2001) *Dynamic equations on time scales: an introduction with applications*. Birkhäuser, Boston
4. Hilscher R, Zeidan V (2004) Calculus of variations on time scales: weak local piecewise C_{rd}^1 solutions with variable endpoints. *J. Math. Anal. Appl.* 289(1):143–166